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Basic Theory of TMD

1. Displacement amplification factor of TMD

To calculate the vibration characteristics of a TMD, a two-story lumped mass model (Figure 1) is considered. Figure 1 assumes a building with a TMD at the top, and consists of a single mass model (main vibration system) and a TMD (secondary vibration system). The main vibration system has mass m_1 stiffness k_1 and damping c_1, and the secondary vibration system has mass m_2, stiffnessk_2, and damping c_2. The relative displacement of the main vibration system to the ground displacement x_0 is x_1, and the relative displacement of the secondary vibration system is x_2.

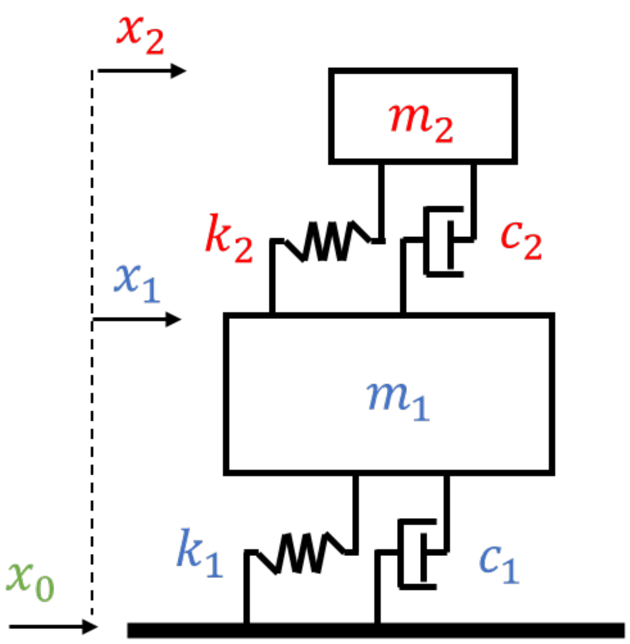


Figure 1

The equation of motion in Figure 1 is as follows

$$\begin{cases} m_1(\ddot{x}_1+\ddot{x}_0)+c_1\dot{x}_1-c_2(\dot{x}_2-\dot{x}_1)+k_1x_1-k_2(x_2-x_1)=0 \\ m_2(\ddot{x}_2+\ddot{x}_0)+c_2(\dot{x}_2-\dot{x}_1)+k_2(x_2-x_1)=0 \end{cases} \tag{1}$$

From Equation (1), treating the ground inertia force as an external force, the following equation of motion is obtained

$$\begin{cases} m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=-m_1\ddot{x}_0 \\ m_2\ddot{x}_2-c_2\dot{x}_1+c_2\dot{x}_2-k_2x_1+k_2x_2=-m_2\ddot{x}_0 \end{cases} \tag{2}$$

Described in matrix form, it is as follows.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = - \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{x}_0 \tag{3}$$

Dividing equations 1 and 2 by m_1 and m_2,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\omega_1 h_1 + 2\mu\omega_2 h_2 & -2\mu\omega_2 h_2 \\ -2\omega_2 h_2 & 2\omega_2 h_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 + \omega_2^2\mu & -\omega_2^2\mu \\ -\omega_2^2 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{x}_0 \tag{4}$$

where ω_1 is the natural circular frequency of the main vibration system, ω_2 is the natural circular frequency of the secondary vibration system, μ is the mass ratio of the main and secondary vibration systems, h_1 is the damping factor of the main vibration system, and h_2 is the damping factor of the secondary vibration system.

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} \ (a) \quad \omega_2 = \sqrt{\frac{k_2}{m_2}} \ (b) \quad \mu = \frac{m_2}{m_1} \ (c) \quad h_1 = \frac{c_1}{2m_1\omega_1} \ (d) \quad h_2 = \frac{c_2}{2m_2\omega_2} \ (e) \tag{5}$$

Here, the ground motions defined by the following equation.

$$x_0 = A_0 e^{i\omega t} \tag{6}$$

The acceleration of the ground motion is then

$$\ddot{x}_0 = -\omega^2 A_0 e^{i\omega t} \tag{7}$$

The displacement, velocity, and acceleration are given as follows

$$x_j = A_j e^{i\omega t} \qquad (j = 0,1,2) \tag{8}$$

$$\dot{x}_j = i\omega A_j e^{i\omega t} \qquad (j = 0,1,2) \tag{9}$$

$$\ddot{x}_j = -\omega^2 A_j e^{i\omega t} \qquad (j = 0,1,2) \tag{10}$$

Substituting Equations (7) through (10) into Equation (4), the following equation is obtained

$$\begin{bmatrix} -\omega^2 + \omega_1^2 + \omega_2^2\mu + (2\omega_1 h_1 + 2\mu\omega_2 h_2)i\omega & -\omega_2^2\mu - 2\mu\omega_2 h_2 i\omega \\ -\omega_2^2 - 2\omega_2 h_2 i\omega & -\omega^2 + \omega_2^2 + 2\omega_2 h_2 i\omega \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} A_0 e^{i\omega t} \tag{11}$$

Dividing Equation (11) by ω_1^2 and transforming the equation, the following equation is obtained

$$\begin{bmatrix} -\beta^2 + 1 + \alpha^2\mu + (2h_1 + 2\mu\alpha h_2)i\beta & -\alpha^2\mu - 2\mu\alpha h_2 i\beta \\ -\alpha^2 - 2\alpha h_2 i\beta & -\beta^2 + \alpha^2 + 2\alpha h_2 i\beta \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} \beta^2 A_0 \\ \beta^2 A_0 \end{Bmatrix} \tag{12}$$

where α is the natural circular frequency ratio of the main vibration system to the secondary vibration system, and β is the natural circular frequency ratio of the main vibration system to the harmonic ground motion.

$$\alpha = \frac{\omega_2}{\omega_1} \quad \beta = \frac{\omega}{\omega_1} \tag{13}$$

The complex amplitudes A_1 and A_2 are obtained as follows.

$$A_1 = \frac{\{\alpha^2(1+\mu)-\beta^2\}+2h_2\alpha\beta(1+\mu)i}{\det \boldsymbol{A}} \times \beta^2 A_0 \tag{14}$$

$$A_2 = \frac{\{1+\alpha^2(1+\mu)-\beta^2\}+2\beta\{h_1+h_2\alpha(1+\mu)\}i}{\det \boldsymbol{A}} \times \beta^2 A_0 \tag{15}$$

Therefore, from Equations (8), (14) and (15), the relative displacements x_1 and x_2 of the main and secondary vibration systems, respectively, are determined as follows

$$x_1 = A_1 e^{i\omega t} = \frac{\beta^2\{\alpha^2(1+\mu)-\beta^2\}+2h_2\alpha\beta^3(1+\mu)i}{\det \boldsymbol{A}} \times A_0 e^{i\omega t} = \frac{R_1+I_1 i}{R_0+I_0 i} \times x_0 \tag{16}$$

$$x_2 = A_2 e^{i\omega t} = \frac{\beta^2\{1+\alpha^2(1+\mu)-\beta^2\}+2\beta^3\{h_1+h_2\alpha(1+\mu)\}i}{\det \boldsymbol{A}} \times A_0 e^{i\omega t} = \frac{R_2+I_2 i}{R_0+I_0 i} \times x_0 \tag{17}$$

where each coefficient is defined as follows

$R_0 = \{(1-\beta^2)(\alpha^2-\beta^2)-\alpha\beta^2(\mu\alpha+4h_1h_2)\}$ (a) $I_0 = 2\beta\{h_1(\alpha^2-\beta^2)+h_2\alpha(1-\beta^2[1+\mu])\}$ (b) (18)

$R_1 = \beta^2\{\alpha^2(1+\mu)-\beta^2\}$ (a) $I_1 = 2h_2\alpha\beta^3(1+\mu)$ (b) (19)

$R_2 = \beta^2\{1+\alpha^2(1+\mu)-\beta^2\}$ (a) $I_2 = 2\beta^3\{h_1+h_2\alpha(1+\mu)\}$ (b) (20)

Equation below is the ratio of the amplitudes of the response of the main vibration system to the pseudo static response x_s, which is called the displacement amplification factor.

$$\left|\frac{x_1}{x_0}\right| = \frac{\sqrt{R_1^2+I_1^2}}{\sqrt{R_0^2+I_0^2}}$$
$$= \frac{\sqrt{\beta^4\{\alpha^2(1+\mu)-\beta^2\}^2+4h_2^2\alpha^2\beta^6(1+\mu)^2}}{\sqrt{\{(1-\beta^2)(\alpha^2-\beta^2)-\alpha\beta^2(\mu\alpha+4h_1h_2)\}^2+4\beta^2\{h_1(\alpha^2-\beta^2)+h_2\alpha(1-\beta^2[1+\mu])\}^2}} \tag{21}$$

$$\left|\frac{x_2}{x_0}\right| = \frac{\sqrt{R_2^2+I_2^2}}{\sqrt{R_0^2+I_0^2}}$$
$$= \frac{\sqrt{\beta^4\{1+\alpha^2(1+\mu)-\beta^2\}^2+4\beta^6\{h_1+h_2\alpha(1+\mu)\}^2}}{\sqrt{\{(1-\beta^2)(\alpha^2-\beta^2)-\alpha\beta^2(\mu\alpha+4h_1h_2)\}^2+4\beta^2\{h_1(\alpha^2-\beta^2)+h_2\alpha(1-\beta^2[1+\mu])\}^2}} \tag{22}$$

Next, consider the case where the displacement of the vibration system is defined in terms of absolute displacement, as shown in Figure 2.

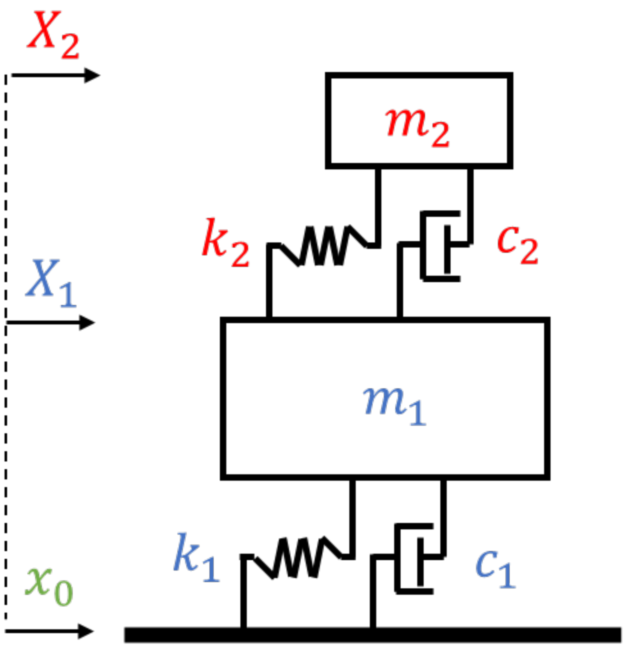


Figure 2

Similar to Equation (2) in the case of relative displacement, the equation of motion when the ground motion $x_0(t)$ is added to the base of the two-story lumped mass model is formulated using the absolute displacements X_1, X_2.

$$\begin{cases} m_1(\ddot{X}_1 - \ddot{x}_0) + (c_1 + c_2)(\dot{X}_1 - \dot{x}_0) - c_2(\dot{X}_2 - \dot{x}_0) + (k_1 + k_2)(X_1 - x_0) - k_2(X_2 - x_0) = -m_1\ddot{x}_0 \\ m_2(\ddot{X}_2 - \ddot{x}_0) - c_2(\dot{X}_1 - \dot{x}_0) + c_2(\dot{X}_2 - \dot{x}_0) - k_2(X_1 - x_0) + k_2(X_2 - x_0) = -m_2\ddot{x}_0 \end{cases} \tag{23}$$

The following equation for the absolute displacement amplification factor can be obtained by performing the same operations as in Equations (4)-(22). The following equation shows that the absolute displacement amplification factor is equal to the complex amplitude of the relative displacement amplification factor plus the complex amplitude of the ground displacement.

$$\left|\frac{X_1}{x_0}\right| = \frac{\sqrt{\{\alpha^2 - \beta^2 - 4\alpha\beta^2 h_1 h_2\}^2 + 4\beta^2\{h_1(\alpha^2 - \beta^2) + \alpha h_2\}^2}}{\sqrt{\{(1 - \beta^2)(\alpha^2 - \beta^2) - \alpha\beta^2(\mu\alpha + 4h_1h_2)\}^2 + 4\beta^2\{h_1(\alpha^2 - \beta^2) + h_2\alpha(1 - \beta^2[1 + \mu])\}^2}}$$
$$= \frac{\sqrt{(R_1 + R_0)^2 + (I_1 + I_0)^2}}{\sqrt{R_0^2 + I_0^2}} \tag{24}$$

$$\left|\frac{X_2}{x_0}\right| = \frac{\sqrt{\{\alpha^2 - 4\alpha\beta^2 h_1 h_2\}^2 + \{2\alpha\beta(\alpha h_1 + h_2)\}^2}}{\sqrt{\{(1-\beta^2)(\alpha^2-\beta^2) - \alpha\beta^2(\mu\alpha + 4h_1h_2)\}^2 + 4\beta^2\{h_1(\alpha^2-\beta^2) + h_2\alpha(1-\beta^2[1+\mu])\}^2}}$$
$$= \frac{\sqrt{(R_2+R_0)^2 + (I_2+I_0)^2}}{\sqrt{R_0^2 + I_0^2}} \quad (25)$$

From Equations (10) and (25) and (26), the acceleration amplification multiplier can be obtained as follows. The following equations show that the ratio of response amplitude to ground motion is equivalent for both displacement and acceleration.

$$\ddot{X}_j(j=1,2) = -\omega^2 A_j e^{i\omega t} = -\omega^2 \times \frac{(R_j+R_0) + (I_j+I_0)i}{R_0 + I_0 i} \times A_0 \times e^{i\omega t} = \frac{(R_j+R_0)+(I_j+I_0)i}{R_0+I_0 i} \times \ddot{x}_0 \quad (26)$$

$$\therefore \left|\frac{\ddot{X}_j}{\ddot{x}_0}\right|(k=1,2) = \frac{\sqrt{(R_j+R_0)^2 + (I_j+I_0)^2}}{\sqrt{R_0^2+I_0^2}} = \left|\frac{X_j}{x_0}\right| \quad (27)$$

The absolute response amplification factor of the main vibration system obtained by Equations (24) is shown in Figure 3. The horizontal axis of the figure is the frequency ratio of the circular frequency (ω) of the harmonic ground motion input to the two-story lumped mass model in Figure 2 normalized by the natural circular frequency (ω_1) of the main vibration system, and the vertical axis is the response of the main vibration system normalized by the amplitude of the ground motion. As shown in Equation (27), the ratio of response amplitude to ground motion is equivalent to both the displacement response and the acceleration response, so they are treated here as a response amplification factor without distinguishing between them.

As shown in Figure 3, the response amplification of the main vibration system has a resonance curve with a valley near a frequency ratio of 1 and two peaks on both sides.

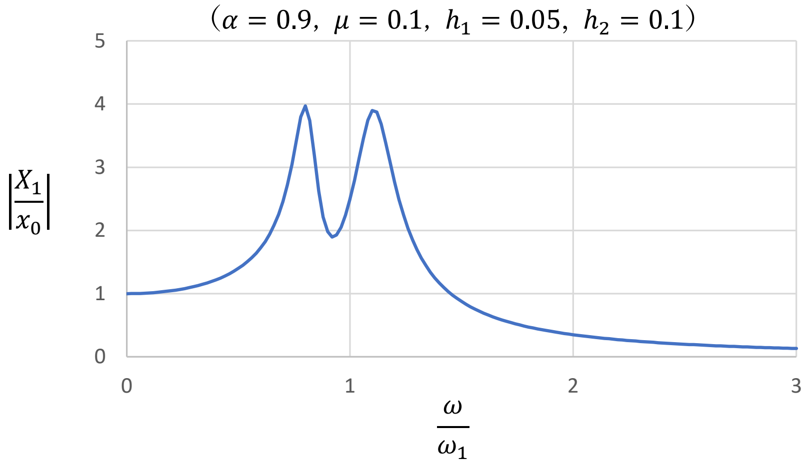
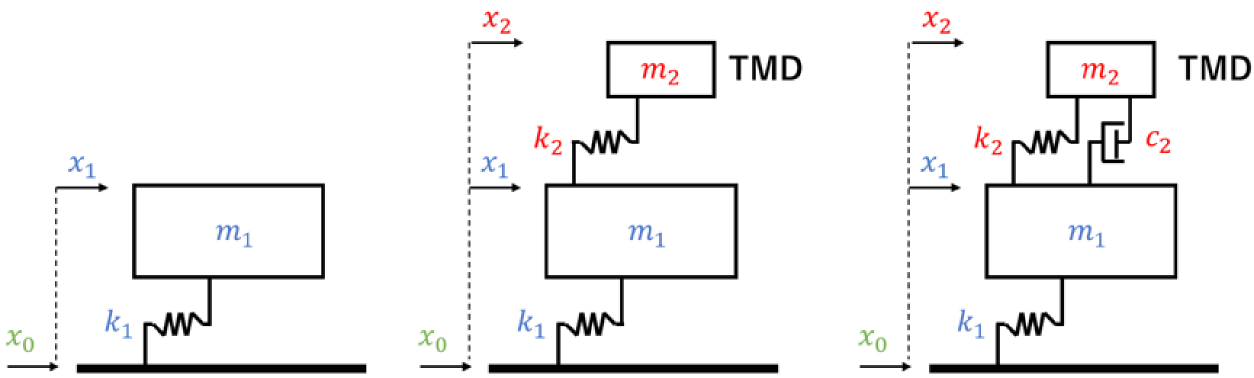


Figure 3 Response amplification factor of the main vibration system

In this section, the optimal values of mass, stiffness, and damping of the TMD are examined from the resonance curve when $h_1 = 0$, without considering damping of the main vibration system. Figure 4(a) shows a model of the main vibration system only, assuming a non-damped building with no TMD installed, Figure 4(b) shows a model of a building with an undamped TMD at the top, and Figure 4(c) shows a model of a building with a damped TMD at the top. Figure 5 shows the resonance curves for each. Here, the frequency ratio $\alpha = 1$, the mass ratio $\mu = 0.05$, and the damping factor $h_2 = 0.05$ for the secondary vibration system is assumed.



(a) No damping　　(b) No damping TMD　　(c) Damping TMD

Figure 4

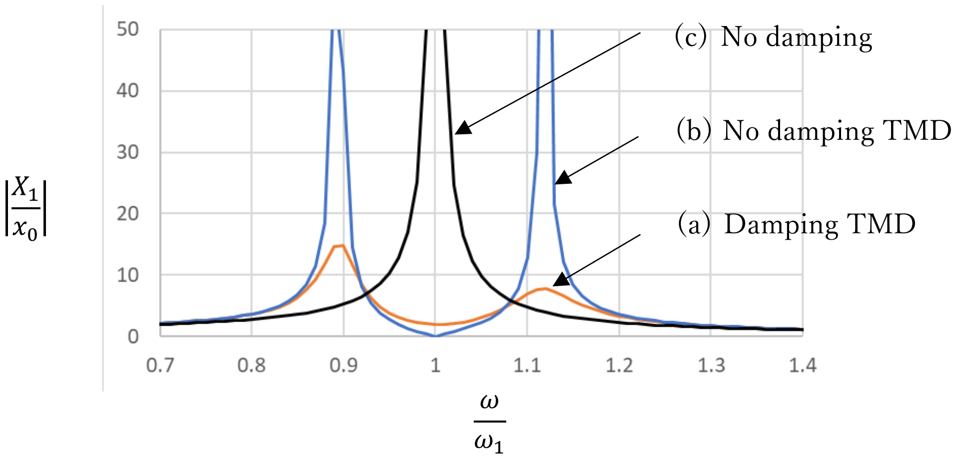


Figure 5 Response amplification factor of the main vibration system

Figure 6 shows the change in response reduction ratio with mass ratio. It can be seen that the response reduction ratio increases as the mass ratio is increased. It can also be seen that not only the amplitude of the maxima is decreasing, but also the interval between the frequencies of the maxima is widening as the mass ratio is increased. This has the effect of preventing the response from increasing when the external force frequency is input in a range slightly off the resonance frequency. Although the installation of large-weight weights can not only increase the response reduction effect but also improve robustness, the mass of the weights should be determined in actual design, taking into consideration the installation in the building, construction restrictions, and safety.

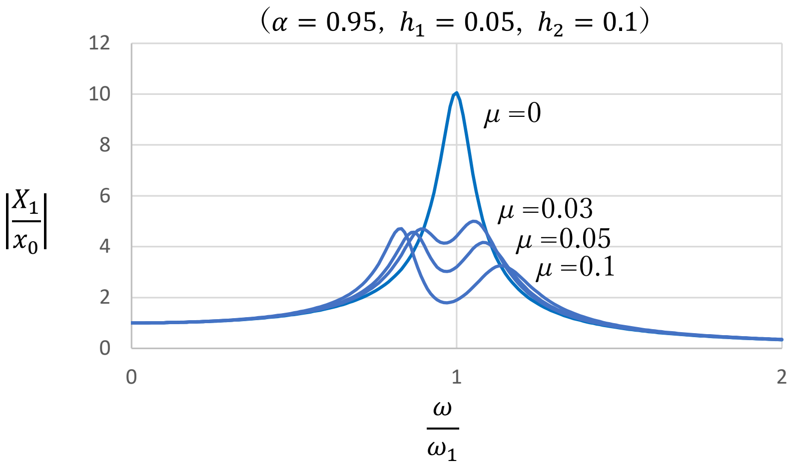


Figure 6 Comparison of resonance curves for different mass ratios

Figure 7 shows the resonance curve for varying the damping factor of the TMD with constant mass ratio and frequency ratio (mass ratio $\mu = 0.1$ and frequency ratio $\alpha = 0.9$). As can be seen, there are two points that always pass through for any damping factor, and these are called fixed points.

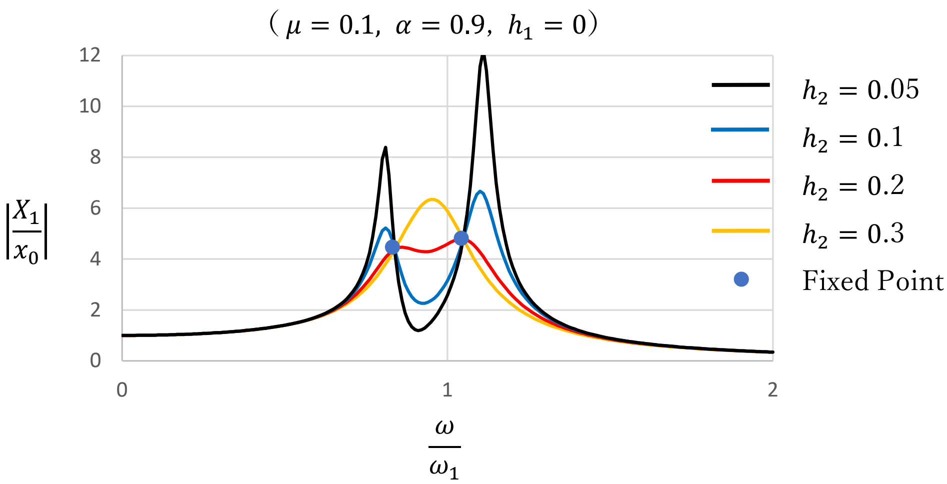


Figure 7 Comparison of resonance curves of different damping factors

The fixed-point theorem defines the optimal value as the frequency ratio at which the two maxima are the same value. When the damping factor is set to the optimum value, a resonance curve with the fixed point as the maximum value is drawn, as shown in Figure 8. This is the resonance curve of the optimum tuned damping that reduces the response factor the most.

$(\mu = 0.1,\ \alpha = 0.909,\ h_1 = 0)$

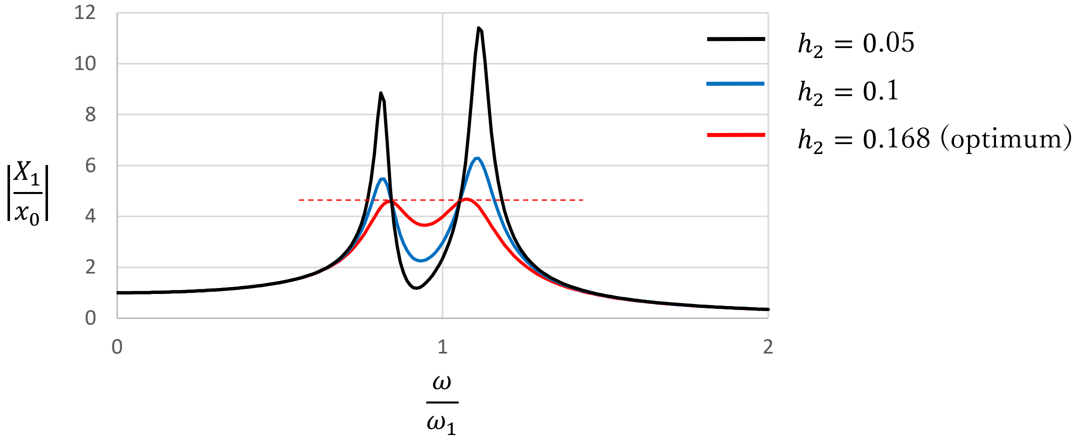


Figure 8 Resonance curve with optimum damping factor

2. Optimum damping factor of TMD

In the absolute displacement amplification factor of Equation (25), if the damping factor of the main vibration system $h_1 = 0$ and the damping factor of the secondary vibration system $h_2 = h$;

$$\left|\frac{X_1}{x_0}\right| = \frac{\sqrt{\{\alpha^2-\beta^2\}^2+4h^2\alpha^2\beta^2}}{\sqrt{\{(1-\beta^2)(\alpha^2-\beta^2)-\mu\alpha^2\beta^2\}^2+4\beta^2\{h\alpha(1-\beta^2[1+\mu])\}^2}}$$
$$= \frac{\sqrt{a_1^2+h^2a_2^2}}{\sqrt{a_3^2+h^2a_4^2}} \tag{29}$$

where

$$a_1 = \alpha^2-\beta^2$$
$$a_2 = 2\alpha\beta$$
$$a_3 = (1-\beta^2)(\alpha^2-\beta^2)-\mu\alpha^2\beta^2$$
$$a_4 = 2\alpha\beta(1-\beta^2[1+\mu]) \tag{30}$$

When $h = 0$

$$\left|\frac{X_1}{x_0}\right| = \left|\frac{a_1}{a_3}\right| \tag{31}$$

When $h = \infty$

$$\left|\frac{X_1}{x_0}\right| = \left|\frac{a_2}{a_4}\right| \tag{32}$$

At the fixed point, since both are equal

$$\left|\frac{a_1}{a_3}\right| = \left|\frac{a_2}{a_4}\right| \tag{33}$$

That is

$$\left(\frac{a_1}{a_3}\right)^2 = \left(\frac{a_2}{a_4}\right)^2 \tag{34}$$

$$\frac{\{\alpha^2-\beta^2\}^2}{\{(1-\beta^2)(\alpha^2-\beta^2)-\mu\alpha^2\beta^2\}^2} = \frac{4\alpha^2\beta^2}{4\beta^2\{\alpha(1-\beta^2[1+\mu])\}^2} = \frac{1}{(1-\beta^2[1+\mu])^2} \tag{35}$$

where

$$a = \alpha^2,\quad b = \beta^2,\quad c = 1+\mu \tag{36}$$

Then

$$\frac{(a-b)^2}{(a-b+b^2-abc)^2} = \frac{1}{(1-bc)^2} \tag{37}$$

$$\frac{(a-b)}{(a-b+b^2-abc)} = -\frac{1}{(1-bc)} \tag{38}$$

$$(1+c)b^2-2(1+ac)b+2a = 0$$

$$b^2 - \frac{2(1+ac)}{1+c}b + \frac{2a}{1+c} = 0 \tag{39}$$

If the solutions of the quadratic equation for b are φ_1 and φ_2, the following equation holds from the relationship between the solutions and coefficients.

$$\varphi_1 + \varphi_2 = \frac{2(1+ac)}{1+c} \tag{40}$$

$$\varphi_1 \varphi_2 = \frac{2a}{1+c} \tag{41}$$

From the condition that the amplitudes at the solutions at φ_1 and φ_2 are equal when the damping factor $h = \infty$, we obtain

$$\left(\frac{a_2}{a_4}\right)^2_{\varphi_1} = \left(\frac{a_2}{a_4}\right)^2_{\varphi_2} \tag{42}$$

$$\frac{1}{(1-\varphi_1 \mathrm{c})^2} = \frac{1}{(1-\varphi_2 \mathrm{c})^2}$$

$$1 - \varphi_1 c = \pm\{1 - \varphi_2 c\}$$

From the condition $\varphi_1 \neq \varphi_2$

$$1 - \varphi_1 c = -\{1 - \varphi_2 c\}$$

$$\varphi_1 + \varphi_2 = \frac{2}{\mathrm{c}} \tag{43}$$

From the relationship between the solutions and coefficients of the quadratic equation

$$\frac{2(1+ac)}{1+c} = \frac{2}{\mathrm{c}} \tag{44}$$

$$a = \frac{1}{c^2} \tag{45}$$

Therefore, **the optimum frequency ratio** is obtained as

$$\frac{\omega_2}{\omega_1} = \frac{1}{1+\mu} \tag{46}$$

The solution of the quadratic equation,

$$b^2 - b\left(\frac{2}{c}\right) + \frac{2a}{1+c} = 0$$

Then

$$b = \frac{\left(\frac{2}{c}\right) \pm \sqrt{\left(\frac{2}{c}\right)^2 - 4\frac{2a}{1+c}}}{2} = \frac{1}{c} \pm \sqrt{\frac{1}{c^2} - \frac{2}{c^2(1+c)}} = \frac{1}{c}\left(1 \pm \sqrt{\frac{c-1}{c+1}}\right) = \frac{1 \pm \sqrt{\frac{\mu}{2+\mu}}}{1+\mu} \tag{47}$$

Since $b = \beta^2 = \left(\frac{\omega}{\omega_1}\right)^2$,

$$\left(\frac{\omega}{\omega_1}\right)_{1,2} = \sqrt{\frac{1 \pm \sqrt{\frac{\mu}{2+\mu}}}{1+\mu}} \tag{48}$$

Next, find the damping factor h such that the response factor is the maximum at the fixed point. From the absolute displacement amplification factor, Equation (29),

$$\left(\frac{X_1}{x_0}\right)^2 = \frac{a_1^2 + h^2 a_2^2}{a_3^2 + h^2 a_4^2} \tag{49}$$

where

$$a_1^2 = \{\alpha^2 - \beta^2\}^2 = (a-b)^2$$

$$a_2^2 = \{2\alpha\beta\}^2 = 4ab$$

$$a_3^2 = \{(1-\beta^2)(\alpha^2-\beta^2) - \mu\alpha^2\beta^2\}^2 = (a - b + b^2 - abc)^2$$

$$a_4^2 = \{2\alpha\beta(1-\beta^2[1+\mu])\}^2 = 4ab(1-bc)^2 \tag{50}$$

Then

$$\left(\frac{X_1}{x_0}\right)^2 = \frac{(a-b)^2 + 4h^2ab}{(a-b+b^2-abc)^2 + 4h^2ab(1-bc)^2} = \frac{f(b)}{g(b)} \tag{51}$$

Since the partial derivative of the equation is zero at the maximum point

$$\frac{\partial}{\partial b}\left\{\left(\frac{X_1}{x_0}\right)^2\right\} = \frac{f'(b)g(b) - f(b)g'(b)}{g(b)^2} = 0 \tag{52}$$

Therefore

$$f'(b)g(b) - f(b)g'(b) = 0 \tag{53}$$

where

$$f(b) = (a-b)^2 + 4h^2ab$$

$$f'(b) = -2(a-b) + 4h^2a$$

$$g(b) = (a-b+b^2-abc)^2 + 4h^2ab(1-bc)^2$$

$$g'(b) = 2(a-b+b^2-abc)(-1+2b-ac) + h^2\{4a(1-bc)^2 - 8abc(1-bc)\}$$

From Equation (37),

$$\frac{(a-b)^2}{(a-b+b^2-abc)^2} = \frac{1}{(1-bc)^2}$$

$$(a-b+b^2-abc)^2 = (a-b)^2(1-bc)^2 \tag{54}$$

$$(a-b+b^2-abc) = -(a-b)(1-bc) \tag{55}$$

Then

$$g(b) = (a-b+b^2-abc)^2 + 4h^2ab(1-bc)^2 = (a-b)^2(1-bc)^2 + 4h^2ab(1-bc)^2$$
$$= (1-bc)^2\{(a-b)^2 + 4h^2ab\}$$

$$g'(b) = 2(a-b+b^2-abc)(-1+2b-ac) + h^2\{4a(1-bc)^2 - 8abc(1-bc)\}$$
$$= 2(a-b)(1-bc)(1-2b+ac) + h^2\{4a(1-bc)^2 - 8abc(1-bc)\}$$

To summarize the above

$$f(b) = (a-b)^2 + 4h^2ab \tag{56}$$

$$f'(b) = -2(a-b) + 4h^2a \tag{57}$$

$$g(b) = (1-bc)^2 f(b) \tag{58}$$

$$g'(b) = (1-bc)[2(a-b)(1-2b+ac) + h^2\{4a(1-bc) - 8abc\}] \tag{59}$$

Substituting into Equation (53),

$$\begin{aligned} &f'(b)g(b) - f(b)g'(b) \\ &\qquad = \{-2(a-b) + 4h^2a\}(1-bc)^2 f(b) \\ &\qquad - f(b)(1-bc)[2(a-b)(1-2b+ac) + h^2\{4a(1-bc) - 8abc\}] = 0 \end{aligned}$$

Dividing by $f(b)(1-bc)$

$$\{-2(a-b) + 4h^2a\}(1-bc) - [2(a-b)(1-2b+ac) + h^2\{4a(1-bc) - 8abc\}] = 0$$

$$\{4a(1-bc) - 4a(1-bc) + 8abc\}h^2 - 2(a-b)\{(1-bc) + (1-2b+ac)\} = 0$$

$$(8abc^3)h^2 - 2(a-b)(2-bc-2b+ac) = 0$$

$$h^2 = \frac{(a-b)(2-bc-2b+ac)}{4abc^3} \tag{60}$$

From Equations (39) and (45)

$$b^2 - \frac{2(1+ac)}{1+c}b + \frac{2a}{1+c} = 0$$

$$a = \frac{1}{c^2}$$

Then

$$b^2 - b\left(\frac{2}{c}\right) + \frac{2a}{1+c} = 0 \quad \rightarrow b^2 - b\left(\frac{2}{c}\right) + \frac{2}{1+c}\frac{1}{c^2} = 0 \quad \rightarrow b^2c^2 - 2bc + \frac{2}{1+c} = 0$$

$$\rightarrow b^2c^2(1+c) - 2bc(1+c) + 2 = 0 \rightarrow b^2c^2 + b^2c^3 - 2bc - 2bc^2 + 2 = 0 \tag{61}$$

Transforming Equation (60)

$$h^2 = \frac{(a-b)(2-bc-2b+ac)}{4abc^3} \rightarrow \frac{\left(\frac{1}{c^2}-b\right)\left(2-bc-2b+\frac{1}{c}\right)}{4bc} \rightarrow \frac{(1-bc^2)\left(2-bc-2b+\frac{1}{c}\right)}{4bc^3} \rightarrow \frac{2-bc-2b+\frac{1}{c}-2bc^2+b^2c^3+2b^2c^2-bc}{4bc^3}$$

$$\rightarrow \frac{b^2c^3-2bc-2bc^2+2-2b+\frac{1}{c}+2b^2c^2}{4bc^3} \rightarrow \frac{-b^2c^2-2b+\frac{1}{c}+2b^2c^2}{4bc^3} \rightarrow \frac{-2b+\frac{1}{c}+b^2c^2}{4bc^3} \rightarrow \frac{-2c+\frac{1}{b}+bc^3}{4c^4}$$

From Equations (36) and (47)

$$c = 1 + \mu$$

$$b = \frac{1 + \sqrt{\frac{\mu}{2+\mu}}}{1+\mu}$$

Substituting these equations,

$$1/b = \frac{1+\mu}{1+\sqrt{\frac{\mu}{2+\mu}}} = \frac{1}{2}(1+\mu)(2+\mu)\left(1-\sqrt{A}\right), \qquad A = \frac{\mu}{2+\mu}$$

$$-2c + \frac{1}{b} + bc^3 = -2(1+\mu) + \frac{1}{2}(1+\mu)(2+\mu)\left(1-\sqrt{A}\right) + (1+\mu)^2\left(1+\sqrt{A}\right) = \frac{(1+\mu)\mu}{2}\left[3+\sqrt{A}\right]$$

Then, Equation (60) will be

$$h^2 = \frac{\mu}{8(1+\mu)^3}\left(3+\sqrt{\frac{\mu}{2+\mu}}\right) \tag{62}$$

In the similar way, when

$$b = \frac{1-\sqrt{\frac{\mu}{2+\mu}}}{1+\mu}$$

The optimum damping factor is

$$h^2 = \frac{\mu}{8(1+\mu)^3}\left(3-\sqrt{\frac{\mu}{2+\mu}}\right) \tag{63}$$

By averaging Equation (62) and (63),

$$h_{opt}^2 = \frac{3\mu}{8(1+\mu)^3} \tag{64}$$

Therefore, **the optimum damping factor** is obtained as

$$h_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)^3}} \tag{65}$$

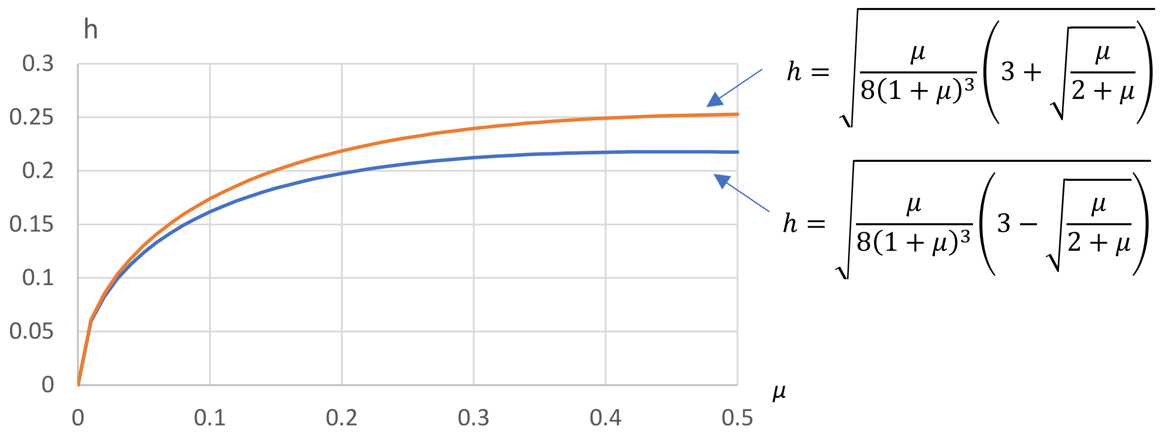


Figure 8 Optimum damping factor